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Glassy Synchronization in a Population of Coupled Oscillators

L. L. Bonilla,[1] C. J. Pérez Vicente,[2] and J. M. Rubí[2]

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A phase model for a population of oscillators with random excitatory and inhibitory mean-field coupling and subject to external white noise random forces is proposed and studied. In the thermodynamic limit different stable phases for the oscillator population may be found: (i) an incoherent state where all possible values of an oscillator phase are equally probable, (ii) a synchronized state where the population has a nonzero collective phase; (iii) a glassy phase where the global synchronization is zero but the oscillators are in phase with the random disorder; and (iv) a mixed phase where the oscillators are partially synchronized and partially in phase with the disorder. These predictions are based upon bifurcation analysis of the reduced equation valid at the thermodynamic limit and confirmed by Brownian simulation.

KEY WORDS: Nonlinear oscillators; synchronization; mean-field model; random disorder model.

1. INTRODUCTION

Aspects of the behavior of many complex systems can be understood by studying the synchronization of large populations of coupled oscillators: dynamics of charge-density waves,(1) chemical reactions,(2) and biological phenomena(3) such as the synchronous flashing of swarms of fireflies(4) and neural network models of sensory processing.(5)

A particularly simple model was put forth by Kuramoto.(6) It consisted of a phase-coupled population of oscillators, each running at a frequency picked up from a given distribution, and all of them coupled by

[1] Escuela Politécnica Superior, Universidad Carlos III de Madrid, 28913 Leganés (Madrid), Spain.

[2] Department de Física Fonamental, Facultat de Física, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain.

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a mean-field interaction. Thus, each oscillator tries to run independently at its own frequency while the coupling tends to synchronize it to all others. When the coupling is sufficiently weak the oscillators run incoherently, whereas beyond a certain threshold collective synchronization is established. In the limit of infinitely many oscillators synchronization is measured by an order parameter which is the average of the cosine of the phase.

The order parameter is different from zero when the oscillators are synchronized. A rigorous analysis of the linear stability of the incoherent (nonsynchronized) state in Kuramoto's model was performed by Strogatz and Mirollo.[7] Because of technical reasons, they added a small independent white noise term to each oscillator. These noise terms can be interpreted as thermal fluctuations (as in the critical dynamics of mean-field ferromagnetic materials[8]) or rapid fluctuations of the intrinsic frequencies of the oscillators. When the distribution $g(\omega)$ of intrinsic oscillator frequencies is unimodal or nonincreasing the synchronization transition is continuous and the synchronized state is stationary.[7] A bimodal $g(\omega)$ may change the character of the transition: it may become a first-order transition to a stationary synchronized state or even a continuous transition to a time-periodic synchronized state.[9]

Kuramoto's model with next-neighbor couplings has been considered by Strogatz and Mirollo,[10] who proved that no synchronization was possible in one-dimensional chains unless the coupling increased as the square root of the number of oscillators. They also proved that in higher-dimensional lattices large clusters of synchronized oscillators necessarily had a spongelike structure.[10] Daido[11] set up a renormalization-group analysis reminiscent of the Kadanoff block spin renormalization group. He defined the "block oscillator" and was able to determine the lower critical dimension for an oscillator ensemble to synchronize. Lumer and Huberman[12] extended Daido's analysis to the case of hierarchical coupling between the oscillators with the branching ratio playing the role of spatial dimensionality. However, all these renormalization group calculations do not describe analytically the synchronized state near the synchronization transition, as was done for a different model with external white noise random processes but without a intrinsic distribution of frequencies.[13]

In this paper we study a generalization of Kuramoto's model whose dynamics is governed by

$$\frac{d\theta_i}{dt} = \omega_i + \gamma_i(t) + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i) \tag{1.1}$$

Here θ_i and ω_i represent the phase and natural frequency of the ith oscillator, K_{ij} the coupling strength between oscillators, and N the size of the population, and $\gamma_i(t)$ are independent white noise random processes with zero mean and correlation

$$\langle \gamma_i(t)\, \gamma_j(t') \rangle = 2D\delta_{ij}\delta(t-t'), \qquad D>0 \tag{1.2}$$

The frequencies ω_i are randomly distributed over the population with a density $g(\omega)$. Often, the coupling strength is considered to be a simply positive constant quantity which is equal for all the oscillators. However, for biological systems[3] the coupling between them may also be random.

This idea was suggested previously by Daido,[14] who considered a site disorder model with Mattis interactions.[15] The most relevant effect found in Daido's model was the appearance of a new glasslike phase reminiscent of random magnetic systems such as spin glasses.[17] However, this model presents two main drawbacks: (i) randomness is removable after a suitable Mattis transformation and (ii) it lacks frustration, which is one of the most important ingredients of random systems.

Motivated by these considerations, we propose a different type of site disorder model characterized by coupling interactions of Van Hemmen's type,[18]

$$K_{ij} = \frac{K_0}{N} + \frac{K_1}{N}\,[\xi_i \eta_j + \xi_j \eta_i], \qquad K_0, K_1 > 0 \tag{1.3}$$

where ξ_i and η_i are independent identically distributed (i.i.d.) random variables that may take values $+1$ or -1 with probability 1/2. (A generalization to i.i.d. random variables with an even distribution about zero and finite variance is straightforward and will be omitted.) Clearly, K_{ij} takes the value K_0/N with probability 1/2 and the values $(K_0 \pm 2K_1)/N$ with probability 1/4. For $K_0 < 2K_1$ we have an excitatory coupling ($K_{ij} > 0$) with probability 3/4 and an inhibitory coupling ($K_{ij} < 0$) with probability 1/4. This is a rough approximation to the Mexican-hat neural coupling[5, 16] or the RKKY interaction in spin glasses.[19]

In contrast to the Mattis interactions considered by Daido, the main effect of the interaction (1.3) is, as in Van Hemmen's model or its soft-spin version,[20] to cause frustration. A positive K_{ij} tends to synchronize the oscillators i, j in phase, whereas a negative coupling tends to place them in phase opposition ($|\theta_i - \theta_j| = \pi$). Our goal is to examine the new properties of the system by evaluating the effect of frustration on the synchronized phase, checking whether other new phases may appear. If this is the case, we will discuss the physical meaning of the new phases in terms of the dynamical behavior of the oscillators.

The study of large assemblies of interacting oscillators is also interesting in other fields, such as neural networks. It has been shown by neurophysiological experiments that neurons in the visual cortex have an oscillatory behavior whose temporal activity can be synchronized by external stimuli.

To reproduce this interesting behavior several mechanisms have been proposed recently. Sompolinsky *et al.*[5] have studied a model in the context of processing visual stimuli coded for orientation whose predictions agree with experimental data obtained from the cat visual cortex. The main features of this model are a generalized version of Kuramoto's equation (1.1) and a Hebb-like coupling

$$K_{ij} = V_i W_{ij} V_j \tag{1.4}$$

where V_i is essentially the firing rate of the ith neuron (oscillator) and W_{ij} specifies the connectivity of the system. For technical reasons Sompolinsky *et al.* had to use only excitatory couplings, although there is evidence that W_{ij} as a function of $|i-j|$ may have an excitatory core and a inhibitory background (see Fig. 4 of ref. 5). In the context of associative memory other authors[21] consider a single oscillator as a system formed by two neurons (actually group of neurons), excitatory and inhibitory, whose dynamics is described by a set of coupled differential equations. In these models inhibition plays an important role since it controls the activity of the elements of the network. In fact, it has been shown that for just two oscillators we may have them either synchronized in phase (excitatory interaction) or in phase opposition shifted 180° (inhibitory interaction). However, it is claimed that phase opposition (the analog of an antiferromagnetic phase) is difficult to achieve for a large population of oscillators. We will show that a mechanism based on Van Hemmen's interaction is capable of displaying such behavior by means of a suitable tuning of the values of the coupling constants K_0, K_1.

Our analysis shows that the model (1.1), (1.3) may have different states in the $N \to \infty$ limit, according to the values of the parameters. These states are characterized by the following order parameters:

$$re^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \tag{1.5}$$

$$q_\xi e^{i\phi_\xi} = \frac{1}{N} \sum_{j=1}^{N} \xi_j e^{i\theta_j} \tag{1.6}$$

$$q_\eta e^{i\phi_\eta} = \frac{1}{N} \sum_{j=1}^{N} \eta_j e^{i\theta_j} \tag{1.7}$$

with r, q_ξ, $q_\eta \geqslant 0$, and are:

1. The incoherent state, $r = q_\xi = q_\eta = 0$.
2. The synchronized state, $r > 0$, $q_\xi = q_\eta = 0$.
3. The glass state, $r = 0$, $q_\xi = q_\eta > 0$.
4. The mixed phase, $r > 0$, $q_\xi = q_\eta > 0$.

The incoherent and synchronized phases also appear in Kuramoto's model,[6,7,9] to which (1.3) reduces when $K_1 = 0$. In the glass phase the oscillators are synchronized to the site disorder but are not globally synchronized. Lastly, in the mixed phase, the oscillators are partly synchronized and partly correlated to the site disorder.

The rest of the paper is organized as follows. In Section 2, we write the nonlinear Fokker–Planck equation for the one-oscillator probability density in the $N \to \infty$ limit. We analyze the incoherent phase and its linear stability in the same section. The bifurcations to the other phases are considered in Section 3. In Section 4 we give the results of a Brownian simulation of the model, which illustrates the dynamical evolution toward the corresponding stable phases of the model at different sides of the phase transition lines. Finally, we devote Section 5 to a discussion of our results and to concluding remarks.

2. PROBABILITY DENSITY AND EVOLUTION

In the limit $N \to \infty$, it is possible to derive a nonlinear Fokker–Planck equation for the one-oscillator probability density. The procedure has been sketched in ref. 22: write $\rho(\theta, t, \omega, \xi, \eta)$ in terms of the N-oscillator-probability-density, ρ_N, solution of the linear Fokker–Planck equation associated to the system (1.1)–(1.3) for an initial condition where ρ_N is the product of the N one-oscillator probability densities. Then write the path integral representation of ρ_N in the resulting expression and perform approximately the integrals by means of the saddle point method in the limit of $N \to \infty$. The resulting expression for $\rho(\theta, t, \omega, \xi, \eta)$ is shown to obey

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta} (\rho v) \tag{2.1}$$

where the drift velocity is given by

$$v(\theta, t, \omega, \xi, \eta) = \omega + K_0 r \sin(\psi - \theta) + K_1 [\xi q_\eta \sin(\phi_\eta - \theta) + \eta q_\xi \sin(\phi_\xi - \theta)] \tag{2.2}$$

and the order parameter amplitudes $r(t)$, $q_\xi(t)$, $q_\eta(t)$ and phases $\psi(t)$, $\phi_\xi(t)$, $\phi_\eta(t)$ are given in terms of ρ and $g(\omega)$ by

$$re^{i\psi} = \int_0^{2\pi} e^{i\theta}\rho(\theta, t, \omega, \xi, \eta)\, g(\omega)\, p(\xi)\, p(\eta)\, d\theta\, d\omega\, d\xi\, d\eta \tag{2.3}$$

$$q_\xi e^{i\phi_\xi} = \int_0^{2\pi} e^{i\theta}\xi\rho(\theta, t, \omega, \xi, \eta)\, g(\omega)\, p(\xi)\, p(\eta)\, d\theta\, d\omega\, d\xi\, d\eta \tag{2.4}$$

and a similar equation for $q_\eta \exp(i\phi_\eta)$. The probability density has to be 2π-periodic in θ and normalized,

$$\int_0^{2\pi} \rho(\theta, t, \omega, \xi, \eta)\, d\theta = 1 \tag{2.5}$$

In Eqs. (2.3)–(2.4) the probability distribution for the random variables ξ and η is given by

$$p(\xi) = \tfrac{1}{2}[\delta(\xi+1) + \delta(\xi-1)] \tag{2.6}$$

Most of our results hold for more general $p(\xi)$ and modifications of the others are straightforward.

A particularly simple stationary solution of (2.1)–(2.6) is the incoherent equiprobability distribution

$$\rho_0(\theta, t, \omega, \xi, \eta) = \frac{1}{2\pi} \tag{2.7}$$

Its linear stability may be analyzed following step by step Strogatz and Mirollo's analysis for Kuramoto's model [to which (2.1)–(2.6) reduce when $K_1 = 0$]. The main results are the following:

Let us consider small disturbances from the incoherent solution

$$\rho(\theta, t, \omega, \xi, \eta) = \frac{1}{2\pi} + \varepsilon e^{\lambda t}\mu(\theta, \omega, \xi, \eta), \qquad \varepsilon \ll 1 \tag{2.8}$$

Then $\mu(\theta, \omega, \xi, \eta)$ obeys the equation

$$\begin{aligned}\lambda\mu = D\frac{\partial^2\mu}{\partial\theta^2} - \omega\frac{\partial\mu}{\partial\theta} + \frac{K_0\tilde{r}}{2\pi}\cos(\tilde{\psi} - \theta)\\ + \frac{K_1}{2\pi}[\tilde{q}_\xi\eta\cos(\tilde{\phi}_\xi - \theta) + \tilde{q}_\eta\xi\cos(\tilde{\phi}_\eta - \theta)]\end{aligned} \tag{2.9}$$

where

$$\tilde{r}e^{i\tilde{\psi}} = \langle e^{i\theta}, \mu(\theta, \omega, \xi, \eta)\rangle$$
$$= \int_0^{2\pi} e^{i\theta}\mu(\theta, \omega, \xi, \eta)\, g(\omega)\, p(\xi)\, p(\eta)\, d\theta\, d\omega\, d\xi\, d\eta \qquad (2.10)$$

$$\tilde{q}_\xi e^{i\tilde{\phi}_\xi} = \langle \xi e^{i\theta}, \mu(\theta, \omega, \xi, \eta)\rangle$$
$$= \int_0^{2\pi} e^{i\theta}\xi\mu(\theta, \omega, \xi, \eta)\, g(\omega)\, p(\xi)\, p(\eta)\, d\theta\, d\omega\, d\xi\, d\eta \qquad (2.11)$$

and similarly for $\tilde{q}_\eta \exp(i\tilde{\phi}_\eta)$. The $\mu(\theta, \omega, \xi, \eta)$ is 2π-periodic in θ and must satisfy

$$\int_0^{2\pi} \mu(\theta, \omega, \xi, \eta)\, d\theta = 0 \qquad (2.12)$$

Let us observe that (a) the continuous spectrum of the operator on the right-hand side of Eq. (2.9) has always $\mathrm{Re}\,\lambda < 0$. The discrete spectrum (when it is nonempty) contains only real eigenvalues when $g(\omega)$ is even and nonincreasing.[7] For a bimodal distribution $g(\omega)$ there may be complex conjugate eigenvalues.[9]

(b) The eigenvalues are determined by noticing that only the first Fourier harmonic of μ

$$\mu(\theta, \omega, \xi, \eta) = \sum_{n=-\infty}^{\infty} c_n(\omega, \xi, \eta) e^{in\theta}, \qquad c_{-n} = \bar{c}_n \qquad (2.13)$$

contributes to the order parameters (2.10) and (2.11). From (2.9) we find

$$c_1 = \frac{1}{\lambda + D + i\omega}\left[\frac{K_0}{2}\int c_1(\omega, \eta, \xi)\, g(\omega)\, d\omega\, p(\eta)\, p(\xi)\, d\eta\, d\xi \right.$$
$$\left. + \frac{K_1}{2}\int c_1(\omega, \eta, \xi)\, g(\omega)\, d\omega\, \eta p(\eta)\, \xi p(\xi)\, d\eta\, d\xi\right] \qquad (2.14)$$

which can be written in terms of the order parameters of the system as

$$c_1 = \frac{1}{\lambda + D + i\omega}\left[\frac{K_0}{4\pi}\tilde{r}e^{i\tilde{\psi}} + \frac{K_1}{4\pi}(q_\xi \eta e^{i\tilde{\phi}_\xi} + q_\eta \xi e^{i\tilde{\phi}_\eta})\right] \qquad (2.15)$$

since

$$\tilde{r}e^{i\tilde{\psi}} = 2\pi \int c_1(t, \omega, \eta, \xi)\, g(\omega)\, d\omega\, p(\eta)\, p(\xi)\, d\eta\, d\xi \qquad (2.16)$$

$$\tilde{q}_\xi e^{i\tilde{\psi}_\xi} = 2\pi \int c_1(t, \omega, \eta, \xi)\, g(\omega)\, d\omega\, \xi p(\eta)\, p(\xi)\, d\eta\, d\xi \qquad (2.17)$$

and a similar expression for $\tilde{q}_\eta \exp(i\tilde{\phi}_\eta)$. As a final result, we find two self-consistent equations for the order parameters which give the critical values of the coupling parameters K_0^c, K_1^c above which the oscillators synchronize:

$$\begin{aligned} \frac{K_0}{2}\left\langle \frac{\lambda+D}{(\lambda+D)^2+\omega^2} \right\rangle &= 1 \\ \left(\frac{K_1}{2}\left\langle \frac{\lambda+D}{(\lambda+D)^2+\omega^2} \right\rangle \right)^2 &= 1 \end{aligned} \tag{2.18}$$

where

$$\left\langle \frac{\lambda+D}{(\lambda+D)^2+\omega^2} \right\rangle = \int g(\omega)\, d\omega \frac{\lambda+D}{(\lambda+D)^2+\omega^2} \tag{2.19}$$

These equations show that the incoherent solution can be unstabilized through two different mechanisms. For $K_0 > K_0^c$ and $K_1 < K_1^c$ the system enters in a synchronized (ferromagnetic) phase which has been extensively analyzed by Kuramoto and by Strogatz and Mirollo. For $K_0 < K_0^c$ and $K_1 > K_1^c$ a new type of entrainment appears characterized by $q_\xi, q_\eta \neq 0$, which means that the oscillators synchronize with the disorder defined in (1.3). If $K_0, K_1 > K_0^c = K_1^c$, then an interesting competitive effect between both phases takes place, leading to a new phase which we describe in the next section.

3. BIFURCATION ANALYSIS

To study the diagram of bifurcations from the incoherent solution to the other stable phases we shall analyze the stationary solutions of the Fokker–Planck equation (2.1)

$$\begin{aligned} 0 = \frac{\partial}{\partial\theta}\bigg(D\frac{\partial\rho}{\partial\theta} - \{\omega + K_0 r \sin(\psi-\theta) \\ + K_1[\xi q_\eta \sin(\phi_\eta - \theta) + \eta q_\xi \sin(\phi_\xi - \theta)]\}\rho \bigg) \end{aligned} \tag{3.1}$$

which has the following solution:

$$\begin{aligned} \rho = AB \exp\bigg\{ \frac{\omega\theta}{D} + \frac{K_0 r}{D}\cos(\psi-\theta) \\ + \frac{K_1}{D}[\xi q_\eta \cos(\phi_\eta-\theta) + \eta q_\xi \cos(\phi_\xi - \theta)] \bigg\} \end{aligned}$$

$$+B\int_0^{\theta} d\theta_1 \exp\left(\frac{\omega(\theta-\theta_1)}{D}+\frac{K_0 r}{D}\left[\cos(\psi-\theta)-\cos(\psi-\theta_1)\right]\right.$$
$$+\frac{K_1}{D}\{\xi q_\eta[\cos(\phi_\eta-\theta)-\cos(\phi_\eta-\theta_1)]$$
$$\left.+\eta q_\xi[\cos(\phi_\xi-\theta)-\cos(\phi_\xi-\theta_1)]\}\right) \tag{3.2}$$

Here A, B are two integration constants which can be determined from normalization and from the periodicity of ρ. We will also assume that $q_\xi = q_\eta = q$ and $\phi_\xi = \phi_\eta = \phi$. This hypothesis has been proved rigorously in other systems with Van Hemmen's interactions[18, 20] and although we will omit any type of proof in this paper, we will check the plausibility of the assumption in the next section just devoted to Brownian simulation. Thus, the final expression for the probability density is

$$\rho=\frac{f(\theta,\xi,\eta)\int_0^{2\pi} d\beta\, h(\theta,\beta,\xi,\eta)}{\int_0^{2\pi} d\theta\, f(\theta,\xi,\eta)\int_0^{2\pi} d\beta\, h(\theta,\beta,\xi,\eta)} \tag{3.3}$$

where

$$f(\theta,\xi,\eta)=\exp\left[\frac{K_0 r}{D}\cos(\psi-\theta)+\frac{K_1}{D}(\eta+\xi)q\cos(\phi-\theta)\right]$$
$$h(\theta,\beta,\xi,\eta)=\exp\left[\frac{-\omega\beta}{D}-\frac{K_0 r}{D}\cos(\psi-\beta-\theta)\right.$$
$$\left.-\frac{K_1 q}{D}(\eta+\xi)\cos(\phi-\beta-\theta)\right]$$

Finally, integration over η, ξ gives the following four equations for the order parameter amplitudes and the mean phases:

$$r=\frac{1}{4}A[\cos(\theta-\psi),1,1]+\frac{1}{2}A[\cos(\theta-\psi),0,0]$$
$$+\frac{1}{4}A[\cos(\theta-\psi),-1,-1] \tag{3.4}$$

$$\tan\psi=\frac{\frac{1}{4}A[\sin\theta,1,1]+\frac{1}{2}A[\sin\theta,0,0]+\frac{1}{4}A[\sin\theta,-1,-1]}{\frac{1}{4}A[\cos\theta,1,1]+\frac{1}{2}A[\cos\theta,0,0]+\frac{1}{4}A[\cos\theta,-1,-1]} \tag{3.5}$$

$$q=\frac{1}{4}A[\cos(\theta-\phi),1,1]-\frac{1}{4}A[\cos(\theta-\phi),-1,-1] \tag{3.6}$$

$$\tan\phi=\frac{A[\sin\theta,1,1]-A[\sin\theta,-1,-1]}{A[\cos\theta,1,1]-A[\cos\theta,-1,-1]} \tag{3.7}$$

where we have defined A as

$$A[x, \xi, \eta] = \int g(\omega)\, d\omega \frac{\int_0^{2\pi} d\theta\, x f(\theta, \xi, \eta) \int_0^{2\pi} d\beta\, h(\theta, \beta, \xi, \eta)}{\int_0^{2\pi} d\theta\, f(\theta, \xi, \eta) \int_0^{2\pi} d\beta\, h(\theta, \beta, \xi, \eta)} \tag{3.8}$$

Numerical integration of these equations for a particular choice of the distribution of frequencies and a given amount of noise D provides the diagram of stable phases as a function of the coupling constants K_0 and K_1 and consequently the dynamical behavior of the whole system.

Our results have been obtained by considering an assembly of oscillators with natural frequencies uniformly distributed on the interval $[-v, v]$ with $v = 0.5$ and for $D = 0.5$. Then, according to (2.18), the incoherent solution is unstable when either K_0 or K_1 is larger than the critical coupling

$$K^c = \frac{2v}{\arctan(v/D)} \tag{3.9}$$

which for the parameters shown previously takes the value $K^c \approx 1.27$. This result is confirmed numerically, as we can see in Fig. 1. For $K_0 > K^c$ and

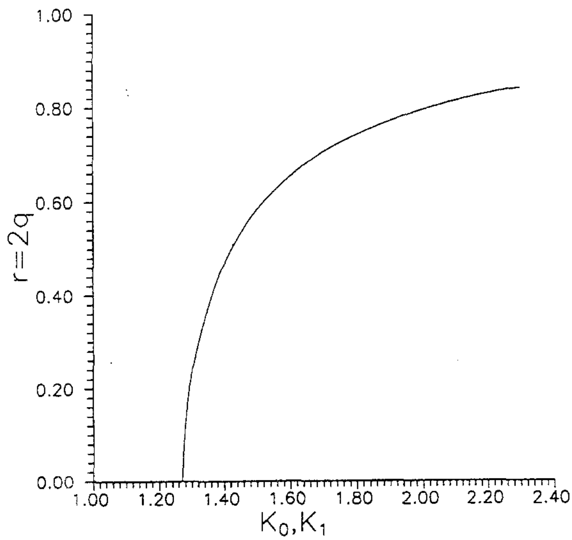

Fig. 1. Order parameters of the system versus the coupling constants for the distribution of natural frequencies and level of noise given in the text. For $K_0, K_1 < K^c$ the only stable solution is the incoherent one characterized by $r = q = 0$. For either (a) $K_0 > K^c$ and $K_1 < K^c$ or (b) $K_0 < K^c$ and $K_1 > K^c$ the incoherent solution is not stable, leading to (a) a synchronized phase where $r \neq 0$ and $q = 0$, or (b) a glassy phase where $r = 0$ and $q \neq 0$. Notice that in case (a) the figure represents r versus K_0, whereas in case (b) the figure represents $2q$ versus K_1. The bifurcation diagram of the system for K_0 and K_1 larger than K^c is shown in Fig. 2.

$K_1 < K^c$ the system enters a phase characterized by order parameters $r \neq 0$ and $q = 0$, which means that a macroscopic fraction of the total number of oscillators (which increases as K_0 increases) is synchronized coherently. In a magnetic system this phase corresponds to a state with nonvanishing magnetization. For $K_0 < K^c$ and $K_1 > K^c$ the system enters a new phase characterized by order parameters $r = 0$ and $q \neq 0$, which means that a macroscopic fraction of the total number of oscillators is in phase with the disorder. This type of entrainment, which we have called glassy synchronization because it is the analogue of the glass phase reminiscent of spin glasses, implies the appearance of clusters of oscillators (spatially disordered) in phase opposition.

Finally, for $K_0 > K^c$ and $K_1 > K^c$ we observe a mixed phase characterized by order parameters $r \neq 0$ and $q \neq 0$ where the oscillators are partially coherently synchronized and partially in phase opposition. It is important to remark that to find the mixed phase from (3.5)–(3.9) it is convenient to take into account the relationship between the phases ϕ and ψ of the two order parameters (2.3) and (2.4); otherwise the system flows toward one of the two branches defined by $r = 0$ and $q \neq 0$ or $r \neq 0$ and $q = 0$. For the numerical values of v and D above, we have found $|\psi - \phi| \approx \pi/2$. Figure 2 illustrates the diagram of stable phases as a function

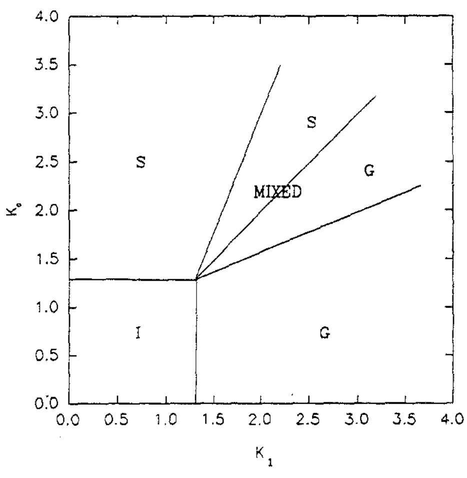

Fig. 2. Schematic phase diagram describing the behavior of the system as a function of the coupling constants K_0 and K_1 for values of the noise D and distribution of frequencies given in the text. There are four phases: Incoherent (I) when both constants K_0 and K_1 are below their critical value. Synchronized (S) and glassy (G) phases are found when K_0 is much bigger than K_1 and vice versa. For intermediate values of the couplings it is possible to find a mixed phase provided the phase difference associated with the order parameters that characterize the system are adequate.

of the coupling constants of the system, keeping constant $D=0.5$ and for the same distribution of frequencies mentioned previously.

To understand the nature of the mixed phase, it is interesting to discuss the following picture introduced by Van Hemmen.[18] Let us consider that our system can be split into two disjoints groups according to the sign of $\xi_i \eta_i$. The points where $\xi_i \eta_i = 1$ are called blue and those with $\xi_i \eta_i = -1$ are called red. Since

$$\xi_i \eta_j + \xi_j \eta_i = \xi_i \eta_j (1 + \xi_i \eta_i \xi_j \eta_j) \tag{3.10}$$

the random part of the interaction defined in (1.3) is only different from 0 for points of the same color and consequently both populations only interact via the systematic term K_0.

Let us imagine for an instant that $K_0 = 0$. Under this condition and applying a Mattis transformation ($e^{i\theta_i} \to \xi_i e^{i\theta_i}$), we can decouple the system into a blue ordered subsystem (ferromagnetic) and a frustrated subsystem (antiferromagnetic).[18] Now, after the new definition of q and r provided by the Mattis transformation the ordered population contributes to q, whereas the red oscillators are uncorrelated and do not contribute to r. Hence, for K_1 large enough ($> K^c$) synchronization described by $q > 0$ and $r = 0$ appears, and we have the glass phase. Further increase of K_1 cannot change r ($= 0$) because there is no interaction between oscillators having different color. Setting $K_0 > 0$ allows for an additional interaction between red and blue oscillators which in turn may provide $r > 0$. Then a mixed phase is obtained.

It is interesting to notice the relevance of the negative interaction (inhibition) for the control of the activity of the oscillators whose temporal correlation can be measured in terms of the different types of synchronization observed. From this point of view a modified version of the system studied in this paper might be of interest for neural network models.

4. BROWNIAN SIMULATION

In this section we want to confirm the results derived in the previous sections through Brownian simulation. We consider a population of 4000 oscillators coupled following (1.3) and with natural frequencies distributed uniformly on the interval $[-0.5, 0.5]$ in 80 groups of 50 elements.

Our results have been obtained by integrating the stochastic equation (1.1) with the Euler method taking as a time step $\Delta t = 0.01$ and a level of noise described by $D = 0.5$. As initial starting point we have assumed for simplicity that all the oscillators are in phase, i.e., $r = 1$, but starting from the incoherent solution ($r = 0$) leads to the same outcomes.

In Figure 3 we observe the evolution of the order parameters of the system for coupling constants $K_0 = 1.5$ and $K_1 = 0$. After a few seconds the system settles into its stationary state, which agrees quantitatively and qualitatively with the results obtained from the numerical integration of the probability density given by (3.3). Notice that our assumption about the equality of both order parameters $q_\eta = q_\xi$ holds for the whole set of simulations we have performed and in particular for those represented by the figures in this paper.

In Fig. 4 we can see the evolution of the system in terms of the order parameters r and $q_\eta = q_\xi$ for couplings $K_0 = 0$ and $K_1 = 1.5$. As expected, a "glass phase" appears which is identified by q values in fair agreement with the theoretical prediction.

Finally, Fig. 5 shows a Brownian simulation describing the mixed phase. The coupling constants are $K_0 = 1.5$ and $K_1 = 1.75$. As one can see, the system needs a lot of time (compared with previous simulations) to reach the stationary state because of the contradictory information that arrives at the oscillators. Again, the simulation confirms the results of Section 3.

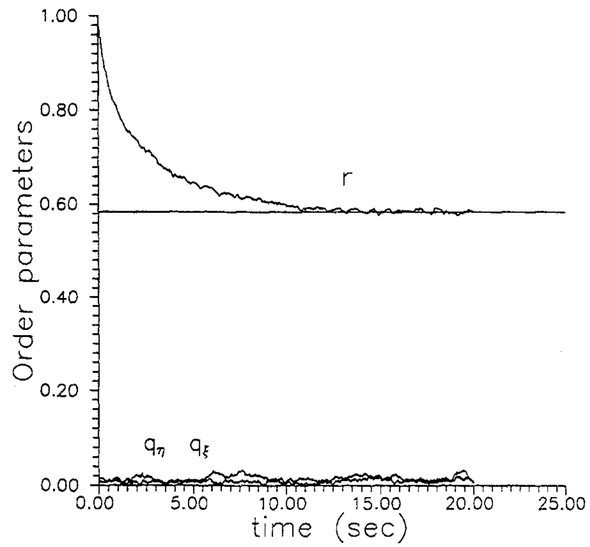

Fig. 3. Brownian simulation describing the temporal evolution of a population of oscillators with intrinsic frequencies uniformly distributed on $[-0.5, 0.5]$ in terms of the order parameters r, q_ξ, q_η, for coupling constants $K_0 = 1.5$ and $K_1 = 0.0$ and a level of noise of $D = 0.5$. As initial condition ($t = 0$) we have assumed that all the oscillators are in phase. For these values a macroscopic fraction of the total number of oscillators synchronize coherently. The straight line shows the theoretical prediction.

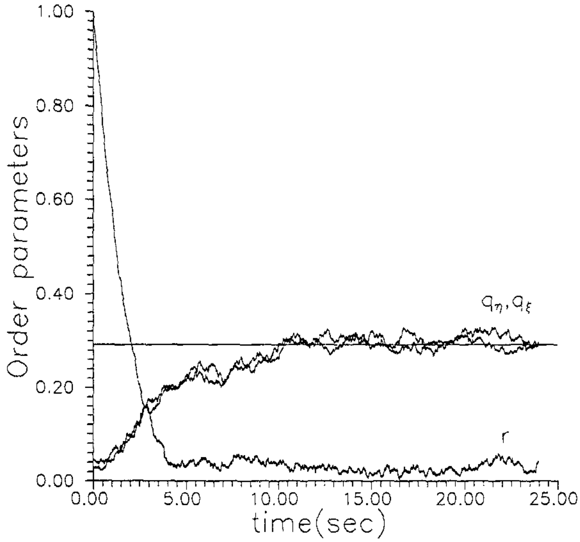

Fig. 4. The same simulation as in Fig. 3 for $K_0 = 0.0$ and $K_1 = 1.5$. The system enters a glassy phase where a macroscopic fraction of the oscillators is correlated with the site disorder. Let us observe that our assumption about the equality of q's is always verified.

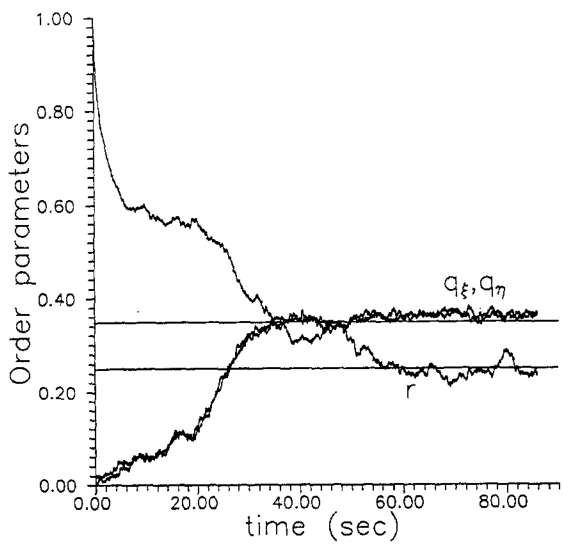

Fig. 5. Results of the Brownian simulation for $K_0 = 1.5$ and $K_1 = 1.75$ and v and D of Fig. 3. As expected, a mixed phase appears characterized by order parameters r, q_ξ, q_η different from zero. Now a group of oscillators is synchronized coherently, whereas another group evolves in phase opposition.

5. DISCUSSION

We have introduced a model of phase-coupled oscillators with random excitatory and inhibitory couplings of Van Hemmen type. New effects include the appearance of stable glassy and mixed phases in which part of the oscillators are frustrated. The effect of couplings is to favor explicit phase differences between pairs of oscillators, but due to the random nature of the interactions these differences cannot be satisfied by all of them. The existence of these new phases and that of the usual stationary synchronized phase has been demonstrated both by means of bifurcation theory and by direct Brownian simulation.

We expect that further new phases will appear when a bimodal distribution of intrinsic oscillators frequencies $g(\omega)$ is used. In ref. 9 it is shown that a new stable phase having a time-periodic order parameter $r(t)$ may appear for a bimodal $g(\omega)$ with widely separated peaks. Besides this phase, when Van Hemmen couplings are used, we expect a glassy phase with time-periodic $q(t)$ for $K_0 < K^c$ and $K_1 > K^c$. As happens in the present paper, it is plausible that for such $g(\omega)$ we may find a mixed phase with time-periodic $r(t)$ and $q(t)$. Since we have now three order parameters at our disposal (r, q, and $\psi - \phi$) and a mechanism to generate two different frequencies, frequency locking, quasiperiodicity, and chaos might appear.

It is also interesting to incorporate Van Hemmen couplings to different models of neural networks. We could have a model with two or more replicas of the present model (with different values of the couplings K_0 and K_1) interacting weakly as in the Sompolinsky *et al.* model of visual perception.[5] This could enable us to describe processes where a group of neural oscillators is synchronized coherently whereas another group displays phase locking with a phase shift of π or, perhaps, glassy or mixed behavior. This could also be an alternative to the mechanism proposed by Wang *et al.*[21] to control the temporal activity of the neurons.

After we submitted this paper we learnt of a work by Daido[23] analyzing the behavior of a population of coupled oscillators with Gaussian random interactions reminiscent of the Sherrington–Kirkpatrick model of spin glasses. We have made a brief and preliminary analysis of the results and have compared them with those of our model. Daido also finds by simulation (he does not present analytical results) a glassy phase characterized by an order parameter identical to our $q_\xi = q_\eta$, although due to the features of his model he never finds synchronized or mixed phases. The most relevant results are the following. First, he finds an algebraic relaxation of the order parameter that characterizes the glassy phase. We think that this effect is due to the absence of noise. In the presence of noise we believe that an exponential relaxation should appear as was shown recently,[24] although the system was completely different. In our model there is no evidence of such algebraic relaxation even for $D=0$. The second point concerns the diffusive motion of the phases. In our preliminary study we have not observed such an effect. These differences are not strange. Although both models introduce frustration into the system, they are intrinsically different and there is no reason to expect the same type of behavior. Looking at these results, it should be interesting to make a more exhaustive comparison of both models, but this deserves a deeper analysis (in preparation).

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